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Laguerre 2D-functions and their application in quantum optics

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Abstract. A set of orthonormalized and complete functions of two real variables in complex representation involving Laguerre polynomials as a substantial part is introduced and is referred to as the set of Laguerre two-dimensional (2D) functions. The properties of the set of Laguerre 2D-functions are discussed and the Fourier and Radon transforms of these functions are calculated. The Laguerre 2D-functions form a basis for a realization of the five-dimensional Lie algebra to the Heisenberg–Weyl group $W(2, \mathrm{R})$ for a two-mode system. Real representation of this set of functions by a sum over products of Hermite functions involving Jacobi polynomials at the zero argument as coefficients is derived and it leads to new connections between Laguerre and Hermite polynomials in both directions. The set of Laguerre 2D-functions is the most appropriate set of functions for the Fock-state representation of quasi-probabilities in quantum optics. The Wigner quasi-probability in Fock-state representation is up to a factor and an argument scaling directly given for each matrix element by a corresponding Laguerre 2D-function. The properties of orthonormality and completeness of the Laguerre 2D-functions provide the Fock-state matrix elements of the density operator directly from the quasi-probabilities. The Peřina–Mišta representation of the Glauber–Sudarshan quasi-probability can be represented with advantage by the Laguerre 2D-functions. The Fock-state matrix elements of the displacement operator and the scalar product of displaced Fock states are closely related to Laguerre 2D-functions.

1. Introduction

We investigate in this paper a two-dimensional (2D) orthonormalized and complete set of functions $l_{m,n}(z, z^*)$ in representation by complex variables ($z = x + \mathrm{i}y$, $z^* = x - \mathrm{i}y$) which involves the Laguerre polynomials $L_n^\nu(u)$ as a substantial part and which we call the set of Laguerre 2D-functions. This is contrary to one-dimensional (1D) sets of Laguerre functions $l_n^\nu(u) = (\mathrm{e}^u u^\nu n!/(n+\nu)!)^{1/2} L_n^\nu(u)$ which are orthonormalized on the positive axis with regard to the indices n, and with ν as fixed parameters. The set of Laguerre 2D-functions was shortly introduced and discussed in [1]. This set of functions is analogous to the 1D orthonormalized and complete set of Hermite functions $h_n(x)$ which involve the Hermite polynomials $H_n(x)$ as a substantial part. It is the most appropriate set of functions for the quasi-probabilities in Fock-state representation, in particular, for the Wigner quasi-probability [2, 3]. We think that the set of Laguerre 2D-functions is appropriate for many problems with functions of two variables due to its completeness, orthonormality and also in classical optics for the representation of Gaussian waves in paraxial approximation. This justifies its introduction and detailed investigation. Contrary to the 2D orthonormalized and

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complete set of functions $h_m(x)h_n(y)$ which is most appropriate in Cartesian coordinates, the set of Laguerre 2D-functions $l_{m,n}(z, z^*)$ is more appropriate for use in polar coordinates.

In the present paper we calculate the Fourier and the Radon transform [4–6] of the set of Laguerre 2D-functions (section 4). The Fourier transform of these functions are again Laguerre 2D-functions. This is very similar to the Hermite functions. The Radon transform of these functions possess the structure of products of two Hermite functions with corresponding indices combined with simple angle functions. In section 5, we derive differential equations for the set of Laguerre 2D-functions which show that they are eigensolutions of the Hamilton operator for a 2D degenerate harmonic oscillator. The Laguerre 2D-functions form a basis for a realization of the 5D Lie algebra to the Heisenberg–Weyl group $W(2, \mathrm{R})$ [7, 8] for a two-mode system and we introduce two annihilation and two creation operators in this realization which act in a simple way onto the Laguerre $2D$-functions. In section 6, we make the transition in the Laguerre 2D-functions to a representation by real variables. We obtain a representation by superposition of products of Hermite functions with Jacobi polynomials taken at the zero argument as coefficients. The connection between the complex and the real representation of the Laguerre 2D-functions leads us to a new identity which, up to now, is only contained in table monographs for the special case of Laguerre polynomials $L_n(x^2 + y^2)$ [9] but not for the associated Laguerre polynomials. The inversion of these relations is also obtained. In section 7, we discuss application of the Laguerre 2D-functions in quantum optics to the Fock-state representation of the quasi-probabilities. In particular, the Wigner quasi-probability in Fock-state representation is up to a factor and to an argument scaling directly given by the whole set of Laguerre 2D-functions with the Fock-state matrix elements as coefficients. The orthonormality and completeness of the Laguerre 2D-functions provides immediately the inversion of the Fock-state matrix elements expressed by the quasi-probabilities. The relation of Laguerre 2D-functions to Hermite functions leads to a new representation of quasi-probabilities. In section 8, we discuss the meaning of the Radon transform of the Wigner quasi-probability which is the main object of quantum tomography [10]. In section 9, we show that the Laguerre 2D-functions are closely related to the Fock-state matrix elements of the displacement operator $D(\alpha, \alpha^*)$ of the Heisenberg–Weyl group $W(1, \mathrm{R})$ [3, 8] and to the scalar product of displaced Fock states. In the appendices we present the calculation of the Radon transform of the Laguerre 2D-functions and give the derivation of relations between real and complex representation of power functions and their inversion involving Jacobi polynomials at zero argument which is applied in section 6.

2. Hermite functions and their application to Fock states

In contrast to Hermite polynomials $H_n(x)$ (Chebyshev 1859, Hermite 1864), the notion of Hermite functions is not often used and is unstable. We introduce it here for a set of orthonormalized functions $h_n(x)$, $(n = 0, 1, 2, \ldots)$ which are the eigenfunctions of the harmonic oscillator in the following way

$$h_n(x) \equiv \frac{1}{\pi^{1/4}} \exp\left(-\frac{x^2}{2}\right) \frac{1}{\sqrt{2^n n!}} H_n(x) \qquad n = 0, 1, 2, \ldots. \tag{2.1}$$

These functions are closely related to the functions of the parabolic cylinder $D_\nu(z)$ by $D_n(\sqrt{2}\,x) = \pi^{1/4}\sqrt{n!}\,h_n(x)$. There is an analogy to the functions which we introduce and investigate in the next sections, however, we will consider here the properties of Hermite functions and some applications in quantum optics. As mentioned, the Hermite functions are orthonormalized and they are complete in the following way

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, h_m(x)h_n(x) = \delta_{m,n} \qquad \sum_{n=0}^{\infty} h_n(x)h_n(y) = \delta(x-y). \tag{2.2}$$

The Fourier transforms $\tilde{h}(u)$ of the Hermite functions are again Hermite functions according to

$$\tilde{h}_n(u) \equiv \int_{-\infty}^{+\infty} \mathrm{d}x\, \exp(-\mathrm{i}ux)h_n(x) = \sqrt{2\pi}(-\mathrm{i})^n h_n(u). \tag{2.3}$$

The Hermite functions form a basis in an infinite-dimensional Hilbert space $\mathcal{H}$ in the realization of this Hilbert space by functions of one variable which is identical with the space L_2 of square-integrable functions. The Hermite functions satisfy the following eigenvalue equation

$$\frac{1}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right)h_n(x) = \left(n + \frac{1}{2}\right)h_n(x) \tag{2.4}$$

that is, the eigenvalue equation for an harmonic oscillator.

The importance of Hermite functions in quantum optics is connected with the property to be the wavefunctions of the eigenstates of the number operator N to eigenvalues n that means of the Fock states $|n\rangle$ in the position and momentum representation as follows ($\hbar$ Planck's constant divided by 2π)

$$\psi(q) \equiv \langle q|n\rangle = \frac{1}{\hbar^{1/4}}h_n\left(\frac{q}{\sqrt{\hbar}}\right) \qquad \psi(p) \equiv \langle p|n\rangle = \frac{(-\mathrm{i})^n}{\hbar^{1/4}}h_n\left(\frac{p}{\sqrt{\hbar}}\right)$$
$$\int_{-\infty}^{+\infty} \mathrm{d}q\, \psi(q)(\psi(q))^* = \int_{-\infty}^{+\infty} \mathrm{d}p\, \psi(p)(\psi(p))^* = 1. \tag{2.5}$$

We now introduce the rotation operator $R(\varphi)$ by (note $[a, a^\dagger] = I, [Q, P] = \mathrm{i}\hbar I$)

$$R(\varphi) \equiv \exp(\mathrm{i}\varphi a^\dagger a) \qquad a \equiv \frac{Q + \mathrm{i}P}{\sqrt{2\hbar}} \qquad a^\dagger \equiv \frac{Q - \mathrm{i}P}{\sqrt{2\hbar}} \tag{2.6}$$

and define the rotated canonical operators $(Q(\varphi), P(\varphi))$ by

$$Q(\varphi) \equiv R(\varphi)Q(R(\varphi))^\dagger = Q\cos\varphi + P\sin\varphi$$
$$P(\varphi) \equiv R(\varphi)P(R(\varphi))^\dagger = -Q\sin\varphi + P\cos\varphi. \tag{2.7}$$

The eigenstates $|q;\varphi\rangle$ of the rotated canonical operator $Q(\varphi)$ to real eigenvalues q can be related to the eigenstates $|q\rangle$ of Q to eigenvalues q in the following way

$$Q(\varphi)|q;\varphi\rangle = q|q;\varphi\rangle \qquad Q|q\rangle = q|q\rangle \qquad |q;\varphi\rangle = R(\varphi)|q\rangle. \tag{2.8}$$

As a generalization of (2.5), we find the expression of $\langle q;\varphi|n\rangle$ by Hermite functions

$$\langle q;\varphi|n\rangle = \langle q|(R(\varphi))^\dagger|n\rangle = \frac{\mathrm{e}^{-\mathrm{i}n\varphi}}{\hbar^{1/4}}h_n\left(\frac{q}{\sqrt{\hbar}}\right). \tag{2.9}$$

The probability densities $\langle q;\varphi|\varrho|q;\varphi\rangle$ in Fock-state representation take on the form

$$\langle q;\varphi|\varrho|q;\varphi\rangle = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle q;\varphi|m\rangle\langle m|\varrho|n\rangle\langle n|q;\varphi\rangle$$
$$= \frac{1}{\sqrt{\hbar}}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle m|\varrho|n\rangle\, \mathrm{e}^{\mathrm{i}(n-m)\varphi}h_m\left(\frac{q}{\sqrt{\hbar}}\right)h_n\left(\frac{q}{\sqrt{\hbar}}\right). \tag{2.10}$$

They are closely related to the Radon transform of the Wigner quasi-probability (section 5).

In the 2D case, one can take the products of Hermite functions

$$h_{m,n}(x, y) \equiv h_m(x)h_n(y) \qquad m, n = 0, 1, 2, \ldots \tag{2.11}$$

as an orthonormalized and complete set of functions. They satisfy the eigenvalue equation for a 2D degenerate harmonic oscillator

$$\frac{1}{2}\left(x^2 - \frac{\partial^2}{\partial x^2} + y^2 - \frac{\partial^2}{\partial y^2}\right) h_m(x)h_n(y) = (m + n + 1)\; h_m(x)h_n(y). \tag{2.12}$$

In comparison to the Laguerre 2D-functions which we define in the next section, the functions $h_m(x)h_n(y)$ could be called Hermite 2D-functions. More general complete sets of functions in the 2D case can be formed by means of the two-variable Hermite polynomials [9, 12–14] as the substantial part.

3. Definition of Laguerre 2D-functions

One substantial part of the Laguerre 2D-functions which we will define are Laguerre polynomials $L_n^\nu(u)$ (Laguerre 1878) which we use in the convention of [9]. We write this definition of Laguerre polynomials here in the following form with complex variables z and z^*

$$\sum_{j=0}^{\{m,n\}} \frac{m!n!}{j!(m-j)!(n-j)!}(-c)^j z^{m-j} z^{*n-j} = n!(-c)^n z^{m-n} L_n^{m-n}\left(\frac{zz^*}{c}\right)$$
$$= m!(-c)^m z^{*\,n-m} L_m^{n-m}\left(\frac{zz^*}{c}\right) \tag{3.1}$$

which is near to the form used in most cases. Note that different definitions of the Laguerre polynomials were used, in particular, before the 1960s. The Laguerre polynomials are the special case $L_n^\nu(u) = ((n+\nu)!/(n!\nu!))\,{}_1F_1(-n, \nu+1; u)$ of the confluent hypergeometric function ${}_1F_1(a, b; u)$.

We now define the following set of Laguerre 2D-functions $l_{m,n}(z, z^*)$ [1]

$$l_{m,n}(z, z^*) \equiv \frac{1}{\sqrt{\pi}} \exp\left(-\frac{zz^*}{2}\right) \frac{1}{\sqrt{m!n!}} \sum_{j=0}^{\{m,n\}} \frac{m!n!}{j!(m-j)!(n-j)!}(-1)^j z^{m-j} z^{*n-j} \tag{3.2}$$

or written by means of the Laguerre polynomials

$$l_{m,n}(z, z^*) = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{zz^*}{2}\right)(-1)^n \sqrt{\frac{n!}{m!}} z^{m-n} L_n^{m-n}(zz^*)$$
$$= \frac{1}{\sqrt{\pi}} \exp\left(-\frac{zz^*}{2}\right)(-1)^m \sqrt{\frac{m!}{n!}} z^{*\,n-m} L_m^{n-m}(zz^*). \tag{3.3}$$

Two alternative definitions of the Laguerre 2D-functions are

$$l_{m,n}(z, z^*) = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{zz^*}{2}\right) \frac{1}{\sqrt{m!n!}} \exp\left(-\frac{\partial^2}{\partial z \partial z^*}\right) z^m z^{*\,n} \tag{3.4}$$

and

$$l_{m,n}(z, z^*) = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{zz^*}{2}\right) \frac{(-1)^{m+n}}{\sqrt{m!n!}} \exp(zz^*) \frac{\partial^{m+n}}{\partial z^{*\,m} \partial z^n} \exp(-zz^*). \tag{3.5}$$

One can check that the explicit calculation of the derivatives in (3.4) and (3.5) leads to the representation (3.2) of the Laguerre 2D-functions.

In polar coordinates, the Laguerre 2D-functions $l_{m,n}(z, z^*)$ possess the following representation

$$l_{m,n}(|z|\,\mathrm{e}^{\mathrm{i}\varphi}, |z|\,\mathrm{e}^{-\mathrm{i}\varphi}) = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{|z|^2}{2}\right) \frac{\mathrm{e}^{\mathrm{i}(m-n)\varphi}}{\sqrt{m!n!}} \sum_{j=0}^{\{m,n\}} \frac{m!n!(-1)^j}{j!(m-j)!(n-j)!} |z|^{m+n-2j}. \tag{3.6}$$

In particular, the angle dependence is given here by the factor $\mathrm{e}^{\mathrm{i}(m-n)\varphi}$ and decompositions into the set of Laguerre 2D-functions are decompositions into a Fourier series with regard to the angle dependence. In the special case $m = n$, one obtains

$$l_{n,n}(z, z^*) = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{zz^*}{2}\right) (-1)^n L_n(zz^*). \tag{3.7}$$

The Laguerre 2D-functions satisfy the following symmetry properties

$$l_{m,n}(z, z^*) = (l_{n,m}(z, z^*))^* = l_{n,m}(z^*, z) \qquad l_{m,n}(-z, -z^*) = (-1)^{m+n} l_{m,n}(z, z^*). \tag{3.8}$$

The most important property of the set of Laguerre 2D-functions is to be orthonormalized in the following way

$$\int \frac{\mathrm{i}}{2}\,\mathrm{d}z \wedge \mathrm{d}z^* (l_{k,l}(z, z^*))^* l_{m,n}(z, z^*) = \delta_{k,m}\delta_{l,n} \tag{3.9}$$

and to satisfy the following completeness relation

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} l_{m,n}(z, z^*)(l_{m,n}(w, w^*))^* = \delta(z - w, z^* - w^*). \tag{3.10}$$

The orthonormality relations can be proved in modified polar coordinates $(u = |z|^2, \varphi)$, where after the first integration over φ there appears a known special integral over products of two Laguerre polynomials combined with exponential and power functions (proof, e.g., in [15]). In a similar way, in polar coordinates for both complex variables z and w, the completeness relation (3.10) can be proved. One can first separate a sum which contains only the moduli $|z|$ and $|w|$ and which can be evaluated by a limiting procedure from a known sum (equation (20) chapter 10.12 in [9], Hille–Hardy or Myller–Lebedeff formulae). Then the remaining sum with the angles as parameters can be evaluated providing the delta function of the difference of the angles.

The orthonormality and the completeness relation in (3.9) and (3.10), involves the product of two Laguerre 2D-functions. We now give two useful relations involving only one Laguerre 2D-function. The first relation is

$$\int \frac{\mathrm{i}}{2}\,\mathrm{d}z \wedge \mathrm{d}z^* \exp\left(\frac{r}{2}zz^*\right) l_{m,n}(z, z^*) = \frac{2\sqrt{\pi}}{1-r}\left(\frac{1+r}{1-r}\right)^n \delta_{m,n} \tag{3.11}$$

which can be proved by accomplishing the integration with the explicit representation of the Laguerre 2D-functions. The second relation is

$$\sum_{n=0}^{\infty} \left(\frac{1+r}{1-r}\right)^n l_{n,n}(z, z^*) = \frac{1-r}{2\sqrt{\pi}} \exp\left(\frac{r}{2}zz^*\right) \tag{3.12}$$

which can be proved by means of the well known generating function of the Laguerre polynomials.

4. Fourier and Radon transform of Laguerre 2D-functions

We now consider two closely related transforms of a function, the Fourier and the Radon transforms [4, 5] and apply this to the Laguerre 2D-functions. We restrict this to the two-dimensional case. The Fourier transform $\tilde{f}(w, w^*)$ of a function $f(z, z^*)$ of two complex variables z and z^* can be defined in the following way

$$\tilde{f}(w, w^*) \equiv \int \frac{\mathrm{i}}{2}\, \mathrm{d}z \wedge \mathrm{d}z^* f(z, z^*) \exp\left\{-\frac{\mathrm{i}}{2}(w^* z + w z^*)\right\} \tag{4.1}$$

and its inversion is determined by

$$f(z, z^*) = \frac{1}{(2\pi)^2} \int \frac{\mathrm{i}}{2}\, \mathrm{d}w \wedge \mathrm{d}w^* \tilde{f}(w, w^*) \exp\left\{\frac{\mathrm{i}}{2}(w^* z + w z^*)\right\}. \tag{4.2}$$

The Radon transform $\breve{f}(w, w^*; c)$ of the function $f(z, z^*)$ in complex representation, we define in the following way (for real representation see [4–6])

$$\breve{f}(w, w^*; c) \equiv \int \frac{\mathrm{i}}{2}\, \mathrm{d}z \wedge \mathrm{d}z^* f(z, z^*) \delta\left\{c - \frac{1}{2}(w^* z + w z^*)\right\} \qquad c \in \mathrm{R} \tag{4.3}$$

where c is an arbitrary real number. The Radon transform $\breve{f}(w, w^*; c)$ depends effectively only on two variables due to the homogeneity

$$|\lambda| \breve{f}(\lambda w, \lambda w^*; \lambda c) = \breve{f}(w, w^*; c) \qquad \lambda \in \mathrm{R} \tag{4.4}$$

where λ is an arbitrary real number. The Radon transform $\breve{f}(w, w^*; c)$ is a real-valued function if $f(z, z^*)$ is a real-valued function and it is here only given in representation by complex variables. The representation by real variables was discussed in [6].

The Radon transform is closely related to the Fourier transform. According to (4.2) by using (4.3), one obtains

$$\tilde{f}(bw, bw^*) = \int_{-\infty}^{+\infty} \mathrm{d}c\, \breve{f}(w, w^*; c) \exp(-\mathrm{i}bc) \qquad b \in \mathrm{R} \tag{4.5}$$

where b is an arbitrary real parameter. This means that one can consider the Radon transform as an intermediate step to the Fourier transform and the Radon transform is in the n-dimensional case 'nearer' to the Fourier transform than to the original function because the last step is a one-dimensional Fourier transformation, whereas the first step includes an $(n-1)$-dimensional integration of the original function. The inversion of this relation provides the Fourier transform in dependence on the Radon transform

$$\breve{f}(w, w^*; c) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}b\, \tilde{f}(bw, bw^*) \exp(\mathrm{i}bc). \tag{4.6}$$

The inversion of the original function from the Radon transform can now be established in a two-step operation with the intermediate determination of the Fourier transform and then with the inversion of the Fourier transform. For this purpose, we need the auxiliary integral

$$\int_{-\infty}^{+\infty} \mathrm{d}r\, |r| \exp(-\mathrm{i}xr) = 2\frac{\partial}{\partial x}\mathcal{R}\left(\frac{1}{x}\right) = -2\mathcal{R}\frac{1}{x^2} \tag{4.7}$$

where $\mathcal{R}$ is the symbol for canonical regularization [16] of a given expression. In particular, canonical regularization of the expressions $1/x$ and $1/x^2$ denote generalized functions determined by the linear functionals [6, 16]

$$\left(\mathcal{R}\frac{1}{x}, \varphi(x)\right) = \int_{-\infty}^{+\infty} \mathrm{d}x\, \mathcal{R}\frac{\varphi(x)}{x} = \int_0^{+\infty} \mathrm{d}x\, \frac{\varphi(x) - \varphi(-x)}{x}$$
$$\left(\mathcal{R}\frac{1}{x^2}, \varphi(x)\right) = \int_{-\infty}^{+\infty} \mathrm{d}x\, \mathcal{R}\frac{\varphi(x)}{x^2} = \int_0^{+\infty} \mathrm{d}x\, \frac{\varphi(x) + \varphi(-x) - 2\varphi(0)}{x^2} \tag{4.8}$$

where $\varphi(x)$ denotes arbitrary basis or multiplicator functions. Canonical regularization of $1/x$ is identical with the principal value of $1/x$. As an example, [6] considers $\varphi(x) = \exp(-x^2)$ which shows that $\mathcal{R}(1/x^2)$ is not a positively definite generalized function. By means of this auxiliary generalized function, the inversion of the primary function from its Radon transform can be represented in the following two forms

$$\begin{aligned} f(z, z^*) &= -\frac{1}{2\pi^2} \int \frac{\mathrm{i}}{2}\, \mathrm{d}w \wedge \mathrm{d}w^* \mathcal{R} \frac{|c|\check{f}(w, w^*; c)}{(c - (1/2)(w^* z + w z^*))^2} \\ &= -\frac{1}{2\pi^2} \int_0^{\pi} \mathrm{d}\varphi \int_{-\infty}^{+\infty} \mathrm{d}c\, \mathcal{R} \frac{|w|^2 \check{f}(|w|\,\mathrm{e}^{\mathrm{i}\varphi}, |w|\,\mathrm{e}^{-\mathrm{i}\varphi}; c)}{(c - (|w|/2)(\mathrm{e}^{-\mathrm{i}\varphi} z + \mathrm{e}^{\mathrm{i}\varphi} z^*))^2}. \end{aligned} \tag{4.9}$$

We now consider the influence of argument transformations in the original function onto the Fourier and Radon transforms. An argument displacement in the original function leads to

$$\begin{aligned} & f(z, z^*) \to f(z - z_0, z^* - z_0^*) \\ &\quad \Leftrightarrow \tilde{f}(w, w^*) \to \exp\left\{-\frac{\mathrm{i}}{2}(w^* z_0 + w z_0^*)\right\} \tilde{f}(w, w^*) \\ &\quad \Leftrightarrow \check{f}(w, w^*; c) \to \check{f}\left(w, w^*; c - \frac{1}{2}(w^* z_0 + w z_0^*)\right). \end{aligned} \tag{4.10}$$

The multiplication of the arguments with a complex number κ leads to the correspondences

$$\begin{aligned} & f(z, z^*) \to f(\kappa z, \kappa^* z^*) \\ &\quad \Leftrightarrow \tilde{f}(w, w^*) \to \frac{1}{\kappa\kappa^*} \tilde{f}\left(\frac{w}{\kappa^*}, \frac{w^*}{\kappa}\right) \\ &\quad \Leftrightarrow \check{f}(w, w^*; c) \to \frac{1}{\kappa\kappa^*} \check{f}\left(\frac{w}{\kappa^*}, \frac{w^*}{\kappa}; c\right) = \check{f}(\kappa w, \kappa^* w^*; \kappa\kappa^* c). \end{aligned} \tag{4.11}$$

More generally, if A is a non-singular matrix ($|A| \neq 0$) which transforms (z, z^*) according to

$$\begin{pmatrix} z' \\ z'^* \end{pmatrix} = A \begin{pmatrix} z \\ z^* \end{pmatrix} = \begin{pmatrix} A_{zz}, & A_{zz^*} \\ A_{z^*z}, & A_{z^*z^*} \end{pmatrix} \begin{pmatrix} z \\ z^* \end{pmatrix} \tag{4.12}$$

then one finds the following correspondences

$$\begin{aligned} & f(z, z^*) \to f(A_{zz} z + A_{zz^*} z^*, A_{z^*z} z + A_{z^*z^*} z^*) \\ &\quad \Leftrightarrow \tilde{f}(w, w^*) \to \frac{1}{|A|} \tilde{f}(w A^{-1}_{z^*z^*} + w^* A^{-1}_{zz^*}, w A^{-1}_{z^*z} + w^* A^{-1}_{zz}) \\ &\quad \Leftrightarrow \check{f}(w, w^*; c) \to \frac{1}{|A|} \check{f}(w A^{-1}_{z^*z^*} + w^* A^{-1}_{zz^*}, w A^{-1}_{z^*z} + w^* A^{-1}_{zz}; c) \\ &\quad = \check{f}(w A_{zz} - w^* A_{zz^*}, -w A_{z^*z} + w^* A_{z^*z^*}; |A| c) \end{aligned} \tag{4.13}$$

with the inverse matrix A^{-1} to A given by

$$A^{-1} = \begin{pmatrix} A^{-1}_{zz}, & A^{-1}_{zz^*} \\ A^{-1}_{z^*z}, & A^{-1}_{z^*z^*} \end{pmatrix} = \frac{1}{|A|} \begin{pmatrix} A_{z^*z^*}, & -A_{zz^*} \\ -A_{z^*z}, & A_{zz} \end{pmatrix}. \tag{4.14}$$

In its quantum-mechanical application to the Radon transform of the Wigner quasi-probability, squeezing transformations of states lead to symplectic transformations of the arguments in the Wigner quasi-probability and corresponding inverse symplectic transformations which are again symplectic transformations in its Fourier and Radon transforms [6].

We now consider the convolution of two functions $g(z, z^*)$ and $h(z, z^*)$ according to

$$f(z, z^*) = g(z, z^*) * h(z, z^*) = \int \frac{\mathrm{i}}{2}\,\mathrm{d}z' \wedge \mathrm{d}z'^* g(z', z'^*) h(z - z', z^* - z'^*). \tag{4.15}$$

The Fourier transform of $f(z, z^*)$ is then the product of the Fourier transforms of $g(z, z^*)$ and $h(z, z^*)$ according to

$$\tilde{f}(w, w^*) = \tilde{g}(w, w^*)\tilde{h}(w, w^*). \tag{4.16}$$

From the connection of the Radon transform with the Fourier transform (4.5) and its inversion (4.6), it follows

$$\begin{aligned} \breve{f}(w, w^*; c) &= \int_{-\infty}^{+\infty} \mathrm{d}c' \int_{-\infty}^{+\infty} \mathrm{d}c''\, \delta(c - c' - c'')\breve{g}(w, w^*; c')\breve{h}(w, w^*; c'') \\ &= \int_{-\infty}^{+\infty} \mathrm{d}c'\, \breve{g}(w, w^*; c')\breve{h}(w, w^*; c - c'). \end{aligned} \tag{4.17}$$

This is a one-dimensional convolution of the Radon transforms $\breve{g}(w, w^*; c)$ and $\breve{h}(w, w^*; c)$ with regard to the variable c.

We now consider the specialization to the Laguerre 2D-functions. The Fourier transforms $\tilde{l}_{m,n}(w, w^*)$ of the Laguerre 2D-functions are again Laguerre 2D-functions according to

$$\begin{aligned} \tilde{l}_{m,n}(w, w^*) &\equiv \int \frac{\mathrm{i}}{2}\,\mathrm{d}z \wedge \mathrm{d}z^* l_{m,n}(z, z^*) \exp\left\{-\frac{\mathrm{i}}{2}(w^* z + w z^*)\right\} \\ &= 2\pi(-\mathrm{i})^{m+n} l_{m,n}(w, w^*). \end{aligned} \tag{4.18}$$

This property is similar to the corresponding property of Hermite functions. It can be proved by using the explicit representation of the Laguerre 2D-functions defined in (3.2). The Radon transform of the Laguerre 2D-functions can be obtained by the transition from the Fourier transform. According to (4.6) and (4.7) one has to evaluate the following integral

$$\breve{l}_{m,n}(w, w^*; c) = (-\mathrm{i})^{m+n} \int_{-\infty}^{+\infty} \mathrm{d}b\, l_{m,n}(bw, bw^*) \exp(\mathrm{i}bc). \tag{4.19}$$

This is made in appendix A. The result has a simple product structure

$$\begin{aligned} \breve{l}_{m,n}(w, w^*; c) &= \sqrt{\frac{2}{ww^*}} \exp\left(-\frac{c^2}{2ww^*}\right) \left(\sqrt{\frac{w}{w^*}}\right)^{m-n} \frac{1}{\sqrt{2^{m+n} m!n!}} \\ &\quad \times H_m\left(\frac{c}{\sqrt{2ww^*}}\right) H_n\left(\frac{c}{\sqrt{2ww^*}}\right) \end{aligned} \tag{4.20}$$

or written by means of the Hermite functions $h_n(x)$ defined in (2.1)

$$\breve{l}_{m,n}(w, w^*; c) = \sqrt{\frac{2\pi}{ww^*}} \left(\sqrt{\frac{w}{w^*}}\right)^{m-n} h_m\left(\frac{c}{\sqrt{2ww^*}}\right) h_n\left(\frac{c}{\sqrt{2ww^*}}\right). \tag{4.21}$$

It is normalized according to

$$\int_{-\infty}^{+\infty} \mathrm{d}c\, \breve{l}_{m,n}(w, w^*; c) = \tilde{l}_{m,n}(0, 0) = \int \frac{\mathrm{i}}{2}\,\mathrm{d}z \wedge \mathrm{d}z^* l_{m,n}(z, z^*) = 2\sqrt{\pi}\delta_{m,n}. \tag{4.22}$$

The structure of the Radon transform of the Laguerre functions as a product of Hermite functions combined with angle functions finds a natural explanation in application to the Wigner quasi-probability and its Fourier and Radon transform (section 8).

5. Differential equations for the Laguerre 2D-functions and Heisenberg–Weyl algebra $w(\mathbf{2},\mathbf{R})$

We now derive the basic differential equations for the set of Laguerre 2D-functions. The starting point is the following known differential equation for the associated Laguerre polynomials $L_n^\nu(u)$ which can be verified by means of the explicit representations

$$\left\{\frac{\partial}{\partial u}u\frac{\partial}{\partial u}+(\nu-u)\frac{\partial}{\partial u}+n\right\}L_n^\nu(u)=0. \tag{5.1}$$

We now introduce modified polar coordinates (u,φ) with the following connections to (z,z^*)

$$z=\sqrt{u}\,\mathrm{e}^{\mathrm{i}\varphi}\qquad z^*=\sqrt{u}\,\mathrm{e}^{-\mathrm{i}\varphi}\qquad u=zz^*\qquad \mathrm{e}^{\mathrm{i}\varphi}=\sqrt{\frac{z}{z^*}}. \tag{5.2}$$

This leads to

$$\begin{aligned}&\frac{1}{z^*}\frac{\partial}{\partial z}=\frac{\partial}{\partial u}-\frac{\mathrm{i}}{2u}\frac{\partial}{\partial\varphi}\qquad \frac{1}{z}\frac{\partial}{\partial z^*}=\frac{\partial}{\partial u}+\frac{\mathrm{i}}{2u}\frac{\partial}{\partial\varphi}\\ &\frac{\partial}{\partial u}=\frac{1}{2}\left(\frac{1}{z^*}\frac{\partial}{\partial z}+\frac{1}{z}\frac{\partial}{\partial z^*}\right)\qquad \frac{1}{u}\frac{\partial}{\partial\varphi}=\mathrm{i}\left(\frac{1}{z^*}\frac{\partial}{\partial z}-\frac{1}{z}\frac{\partial}{\partial z^*}\right)\end{aligned} \tag{5.3}$$

and to the following operator

$$\frac{\partial^2}{\partial z\partial z^*}=\frac{\partial}{\partial u}u\frac{\partial}{\partial u}+\frac{1}{4u}\frac{\partial^2}{\partial\varphi^2}. \tag{5.4}$$

The differential equation (5.1) can now be represented in the following form

$$\left\{\frac{\partial^2}{\partial z\partial z^*}+\frac{m-n-zz^*}{2}\left(\frac{1}{z^*}\frac{\partial}{\partial z}+\frac{1}{z}\frac{\partial}{\partial z^*}\right)+n\right\}L_n^{m-n}(zz^*)=0 \tag{5.5}$$

and, additionally, the Laguerre polynomials $L_n^{m-n}(zz^*)$ satisfy the equation

$$\left(\frac{1}{z^*}\frac{\partial}{\partial z}-\frac{1}{z}\frac{\partial}{\partial z^*}\right)L_n^{m-n}(zz^*)=0 \tag{5.6}$$

due to the independence of $L_n^{m-n}(zz^*)$ on the angle φ. By using this, one finds the following differential equation for the Laguerre 2D-functions

$$\left\{\frac{\partial^2}{\partial z\partial z^*}-\frac{zz^*}{4}+\frac{m+n+1}{2}\right\}l_{m,n}(z,z^*)=0. \tag{5.7}$$

Furthermore, one finds that they satisfy the following eigenvalue equation for the operator $-\mathrm{i}\partial/\partial\varphi$ expressed by the variable (z,z^*)

$$\left(z\frac{\partial}{\partial z}-z^*\frac{\partial}{\partial z^*}\right)l_{m,n}(z,z^*)=(m-n)\,l_{m,n}(z,z^*). \tag{5.8}$$

One can write equation (5.7) as the following eigenvalue equation

$$\left(\frac{zz^*}{2}-2\frac{\partial^2}{\partial z\partial z^*}\right)l_{m,n}(z,z^*)=(m+n+1)\,l_{m,n}(z,z^*). \tag{5.9}$$

The operator on the left-hand side is the Hamiltonian operator for a degenerate 2D harmonic oscillator with frequencies $\omega_1=\omega_2\equiv\omega$ divided by $\hbar\omega$. The Laguerre 2D-functions $l_{m,n}(z,z^*)$ are eigensolutions of this operator to eigenvalues $(m+n+1)$. This operator alone cannot discriminate with regard to its eigensolutions between different m and n but only between different sums $(m+n)$. One can therefore form linear combinations of the

Laguerre 2D-functions $l_{m,n}(z, z^*)$ with equal sum $(m+n)$ which are also eigensolutions of this operator to the same eigenvalue. A full discrimination between different m and n can be obtained if, additionally to (5.9) we use the eigenvalue equation (5.8) for the operator $-\mathrm{i}\partial/\partial\varphi$ expressed by (z, z^*). Due to this degeneracy, one can consider combinations of the eigenvalue equations (5.8) and (5.9) and obtain in this way other eigenvalue equations for the Laguerre 2D-functions.

One can introduce more basic operators for the Laguerre 2D-functions such as annihilation and creation operators. By using the explicit representations for the Laguerre 2D-functions, one proves (abstract representation → concrete realization)

$$\begin{aligned} a_+ l_{m,n} &= \sqrt{m}\, l_{m-1,n} \rightarrow \left(\frac{z^*}{2} + \frac{\partial}{\partial z}\right) l_{m,n}(z, z^*) = \sqrt{m}\, l_{m-1,n}(z, z^*) \\ a_- l_{m,n} &= \sqrt{n}\, l_{m,n-1} \rightarrow \left(\frac{z}{2} + \frac{\partial}{\partial z^*}\right) l_{m,n}(z, z^*) = \sqrt{n}\, l_{m,n-1}(z, z^*) \end{aligned} \tag{5.10}$$

as well as

$$\begin{aligned} a_+^\dagger l_{m,n} &= \sqrt{m+1}\, l_{m+1,n} \rightarrow \left(\frac{z}{2} - \frac{\partial}{\partial z^*}\right) l_{m,n}(z, z^*) = \sqrt{m+1}\, l_{m+1,n}(z, z^*) \\ a_-^\dagger l_{m,n} &= \sqrt{n+1}\, l_{m,n+1} \rightarrow \left(\frac{z^*}{2} - \frac{\partial}{\partial z}\right) l_{m,n}(z, z^*) = \sqrt{n+1}\, l_{m,n+1}(z, z^*). \end{aligned} \tag{5.11}$$

The introduced operators satisfy the commutation relations

$$\begin{aligned} [a_+, a_+^\dagger] &= I \rightarrow \left[\frac{z^*}{2} + \frac{\partial}{\partial z}, \frac{z}{2} - \frac{\partial}{\partial z^*}\right] = 1 \\ [a_-, a_-^\dagger] &= I \rightarrow \left[\frac{z}{2} + \frac{\partial}{\partial z^*}, \frac{z^*}{2} - \frac{\partial}{\partial z}\right] = 1 \\ [a_+, a_-^\dagger] &= 0 \rightarrow \left[\frac{z^*}{2} + \frac{\partial}{\partial z}, \frac{z^*}{2} - \frac{\partial}{\partial z}\right] = 0 \\ [a_-, a_+^\dagger] &= 0 \rightarrow \left[\frac{z}{2} + \frac{\partial}{\partial z^*}, \frac{z}{2} - \frac{\partial}{\partial z^*}\right] = 0 \\ [a_+, a_-] &= 0 \rightarrow \left[\frac{z^*}{2} + \frac{\partial}{\partial z}, \frac{z}{2} + \frac{\partial}{\partial z^*}\right] = 0 \\ [a_+^\dagger, a_-^\dagger] &= 0 \rightarrow \left[\frac{z}{2} - \frac{\partial}{\partial z^*}, \frac{z^*}{2} - \frac{\partial}{\partial z}\right] = 0. \end{aligned} \tag{5.12}$$

These relations show that $z^*/2 + \partial/\partial z$ and $z/2 + \partial/\partial z^*$ are annihilation operators for the Laguerre 2D-functions which reduce one of the indices in unit steps, and $z/2 - \partial/\partial z^*$ and $z^*/2 - \partial/\partial z$ are creation operators which increase one of the indices in unit steps. The relations (5.12) imply that the 5 operators $a_+ \rightarrow z^*/2 + \partial/\partial z$, $a_+^\dagger \rightarrow z/2 - \partial/\partial z^*$, $a_- \rightarrow z/2 + \partial/\partial z^*$, $a_-^\dagger \rightarrow z^*/2 - \partial/\partial z$, $I \rightarrow 1$ are closed with regard to the commutation relations. Therefore, they form a realization of an abstract 5D Lie algebra by multiplication and differentiation operators which is the Lie algebra to the Heisenberg–Weyl group $W(2, \mathrm{R})$ or to its complex extension $W(2, \mathrm{C})$ for a two-mode system and the Laguerre 2D-functions form a certain basis for this realization of the Heisenberg–Weyl algebra $w(2, \mathrm{R})$ or for its complex extension $w(2, \mathrm{C})$ [7] (see [8] for notion of Heisenberg–Weyl algebra and group).

The operator of the eigenvalue equation (5.9) can be formed by the basis operators of this Heisenberg–Weyl algebra as follows

$$\frac{1}{2}(a_+a_+^\dagger + a_+^\dagger a_+ + a_-a_-^\dagger + a_-^\dagger a_-)$$
$$\to \frac{1}{2}\left\{\left(\frac{z^*}{2}+\frac{\partial}{\partial z}\right)\left(\frac{z}{2}-\frac{\partial}{\partial z^*}\right)+\left(\frac{z}{2}-\frac{\partial}{\partial z^*}\right)\left(\frac{z^*}{2}+\frac{\partial}{\partial z}\right)\right.$$
$$\left.+\left(\frac{z}{2}+\frac{\partial}{\partial z^*}\right)\left(\frac{z^*}{2}-\frac{\partial}{\partial z}\right)+\left(\frac{z^*}{2}-\frac{\partial}{\partial z}\right)\left(\frac{z}{2}+\frac{\partial}{\partial z^*}\right)\right\} = \frac{zz^*}{2} - 2\frac{\partial^2}{\partial z\partial z^*} \tag{5.13}$$

and, analogously, the operator of equation (5.8)

$$a_+a_+^\dagger - a_-a_-^\dagger \to z\frac{\partial}{\partial z} - z^*\frac{\partial}{\partial z^*}. \tag{5.14}$$

These operators are only two of ten independent operators which can be formed by quadratic combinations of the basic operators given in (5.10) and (5.11) and which form the Lie algebra $sp(4,\mathrm{R})$ to the symplectic group $Sp(4,\mathrm{R})$ in the 4D phase space [17] or its complex extension $Sp(4,\mathrm{C})$. This group contains interesting subgroups as, for example, the 3D group $SU(2)$ and 'physically' different groups $SU(1,1)$ (for each of the two modes and for mixing of the two modes). The group $SU(2)$ is the group of the transformations of the two modes by lossless beam splitting [18]. The different groups $SU(1,1)$ are groups of squeezing transformations of each of the modes separately or of two-mode squeezing, correspondingly [17]. If one uses the Laguerre 2D-functions as basis functions of the 2D harmonic oscillator, the mentioned subgroups of $Sp(4,\mathrm{R})$ and the whole group $Sp(4,\mathrm{R})$ generate transformations of these basis functions which we, however, do not investigate in the present paper.

By using both the eigenvalue equations (5.8) and (5.9), one can prove in a standard way that the Laguerre 2D-functions $l_{k,l}(z,z^*)$ and $l_{m,n}(z,z^*)$ are orthogonal to each other for $k \neq m$ or $l \neq n$ in the sense of equation (3.9). The value of the integral for $k = m$ and $l = n$ and therefore the normalization has to be calculated then in a direct way by using the definition of the Laguerre 2D-functions with the result given in (3.9).

6. Real representation of Laguerre 2D-functions

We now derive the real representation of Laguerre 2D-functions. The first step is to obtain the real representation of powers $z^m z^{*n}$ in the following way (derivation in appendix B)

$$(x+\mathrm{i}y)^m(x-\mathrm{i}y)^n = \sum_{j=0}^{m+n}(\mathrm{i}2)^j P_j^{(m-j,n-j)}(0)x^{m+n-j}y^j \tag{6.1}$$

where $P_j^{(k,l)}(u)$ denotes the Jacobi polynomials (Jacobi 1859). The next step is to write the form (3.4) of the definition of the Laguerre 2D-functions by real variables and to apply (6.1) that leads to

$$l_{m,n}(x+\mathrm{i}y, x-\mathrm{i}y) = \frac{1}{\sqrt{\pi}}\exp\left(-\frac{x^2+y^2}{2}\right)\frac{1}{\sqrt{m!n!}}$$
$$\times \exp\left\{-\frac{1}{4}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\right\}(x+\mathrm{i}y)^m(x-\mathrm{i}y)^n$$

$$= \frac{1}{\sqrt{\pi}} \exp\left(-\frac{x^2+y^2}{2}\right) \frac{1}{\sqrt{m!n!}} \exp\left\{-\frac{1}{4}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\right\}$$
$$\times \sum_{j=0}^{m+n} (\mathrm{i}2)^j P_j^{(m-j,n-j)}(0) x^{m+n-j} y^j. \tag{6.2}$$

We now use the following representation of the Hermite polynomials [19] (found earlier in [20])

$$H_n(x) = \exp\left(-\frac{1}{4}\frac{\partial^2}{\partial x^2}\right)(2x)^n = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{k!(n-2k)!}(2x)^{n-2k} \tag{6.3}$$

which can be checked immediately by explicit calculation of the derivatives after Taylor series expansion of the exponential (convolution) operator applied to the asymptotics of the Hermite polynomials. By using this, we obtain the following real representation of the Laguerre 2D-functions

$$l_{m,n}(x+\mathrm{i}y, x-\mathrm{i}y) = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{x^2+y^2}{2}\right) \frac{1}{2^{m+n}\sqrt{m!n!}}$$
$$\times \sum_{j=0}^{m+n} (\mathrm{i}2)^j P_j^{(m-j,n-j)}(0) H_{m+n-j}(x) H_j(y). \tag{6.4}$$

This can be represented by means of the Hermite functions in the following way

$$l_{m,n}(x+\mathrm{i}y, x-\mathrm{i}y) = \frac{1}{\sqrt{2^{m+n}m!n!}} \sum_{j=0}^{m+n} (\mathrm{i}2)^j P_j^{(m-j,n-j)}(0) \sqrt{(m+n-j)!j!}\, h_{m+n-j}(x) h_j(y) \tag{6.5}$$

and the inversion of this relation by analogy to (B.3) and (B.4) leads to

$$h_m(x)h_n(y) = \frac{(-\mathrm{i})^n}{\sqrt{2^{m+n}m!n!}} \sum_{j=0}^{m+n} 2^j P_j^{(m-j,n-j)}(0) \sqrt{(m+n-j)!j!}\, l_{m+n-j,j}(x+\mathrm{i}y, x-\mathrm{i}y). \tag{6.6}$$

Thus, we have obtained the real representation of the Laguerre 2D-functions by superposition of Hermite 2D-functions and its inversion. These relations show again that Laguerre 2D-functions $l_{m,n}(z, z^*)$ as well as Hermite 2D-functions $h_m(x)h_n(y)$ are eigensolutions of the two-dimensional degenerate harmonic oscillator to eigenvalues $(m+n+1)$ and only a further eigenvalue equation can discriminate between different m and n.

The values of the Jacobi polynomials taken for zero argument $2^j P_j^{(m-j,n-j)}(0)$ are (positive or negative) integers. In the general case, they are not representable by a closed formula of the multiplicative type. We have checked this by decomposition into prime factors by means of a computer, where for small numbers of (j, m, n) there appear, in an irregular way, high prime factors which make the existence of a closed formula of the mentioned type impossible. However, in the case of equal upper indices, we have the following simple representation for even lower indices (binomial coefficients) and odd lower indices (vanishing) obtained from relations to certain Gegenbauer and Legendre polynomials [14]

$$P_{2k}^{(l,l)}(0) = \frac{(-1)^k(2k+l)!}{2^{2k}k!(k+l)!} \qquad P_{2k+1}^{(l,l)}(0) = 0 \Rightarrow (\mathrm{i}2)^{2k} P_{2k}^{(n-2k,n-2k)}(0) = \frac{n!}{k!(n-k)!} \tag{6.7}$$

for arbitrary integer k. This means that in case of $m = n$, the Laguerre 2D-functions possess the following relation to products of Hermite functions

$$l_{n,n}(x + \mathrm{i}y, x - \mathrm{i}y) = \frac{1}{2^n n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \sqrt{(2n-2k)!(2k)!}\, h_{2n-2k}(x) h_{2k}(y) \tag{6.8}$$

and products of Hermite functions of the same index can be represented by Laguerre 2D-functions according to

$$h_n(x)h_n(y) = \frac{(-\mathrm{i})^n}{2^n n!} \sum_{k=0}^{n} \frac{(-1)^k n!}{k!(n-k)!} \sqrt{(2n-2k)!(2k)!}\, l_{2n-2k,2k}(x + \mathrm{i}y, x - \mathrm{i}y). \tag{6.9}$$

By comparison of (6.8) with (3.7) and (2.1), one obtains the identity

$$L_n(x^2 + y^2) = \frac{(-1)^n}{2^{2n} n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} H_{2n-2k}(x) H_{2k}(y). \tag{6.10}$$

This is a known identity [9] (factor $1/2^{2n}$ is absent there). The generalization of this identity taken from (3.3) and (6.5) is

$$\begin{aligned}(-1)^n n!(x + \mathrm{i}y)^{m-n} L_n^{m-n}(x^2 + y^2) &= (-1)^m m!(x - \mathrm{i}y)^{n-m} L_m^{n-m}(x^2 + y^2)\\ &= \frac{1}{2^{m+n}} \sum_{j=0}^{m+n} (\mathrm{i}2)^j P_j^{(m-j,n-j)}(0) H_{m+n-j}(x) H_j(y).\end{aligned} \tag{6.11}$$

The known special case where $m = n$ given in (6.10) involves only the even Hermite polynomials on the right-hand side, whereas we have here involved all Hermite polynomials from index 0 on up to the maximal index $(m+n)$. From the inversion of the relation (6.5) in (6.6) it follows if we use the definition of the Hermite functions in (2.1) and of the Laguerre 2D-functions in (3.2) that

$$\begin{aligned}H_m(x)H_n(y) &= (-\mathrm{i})^n \sum_{j=0}^{m+n} (-2)^j P_j^{(m-j,n-j)}(0) j!(x + \mathrm{i}y)^{m+n-2j} L_j^{m+n-2j}(x^2 + y^2)\\ &= \mathrm{i}^n (-1)^m \sum_{j=0}^{m+n} (-2)^j P_j^{(m-j,n-j)}(0)\\ &\quad \times (m+n-j)!(x - \mathrm{i}y)^{2j-m-n} L_{m+n-j}^{2j-m-n}(x^2 + y^2).\end{aligned} \tag{6.12}$$

In the special case $m = n$ by using (6.7) this leads to the following identity

$$\begin{aligned}H_n(x)H_n(y) &= (-\mathrm{i})^n \sum_{k=0}^{n} \frac{(-1)^k n!}{k!(n-k)!} (2k)!(x + \mathrm{i}y)^{2(n-2k)} L_{2k}^{2(n-2k)}(x^2 + y^2)\\ &= (-\mathrm{i})^n \sum_{k=0}^{n} \frac{(-1)^k n!}{k!(n-k)!} (2n-2k)!(x - \mathrm{i}y)^{2(2k-n)} L_{2n-2k}^{2(2k-n)}(x^2 + y^2).\end{aligned} \tag{6.13}$$

which can be also obtained from (6.9).

By multiplication of (6.5) with $(l_{k,l}(x+\mathrm{i}y, x-\mathrm{i}y))^*$, integration over the two-dimensional space of coordinates (x, y) and by using the orthonormality of the Laguerre 2D-functions and of the Hermite functions, one obtains the following identity checked by computer

$$\frac{1}{2^{m+n} m! n!} \sum_{j=0}^{m+n} 4^j (m+n-j)! j! P_j^{(m+n-l-j,l-j)}(0) P_j^{(m-j,n-j)}(0) = \delta_{l,n} \tag{6.14}$$

independently of the integer value m.

The two differential equations (5.8) and (5.9) for the Laguerre 2D-functions are

$$\mathrm{i}\left(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right)l_{m,n}(x+\mathrm{i}y,x-\mathrm{i}y)=(m-n)l_{m,n}(x+\mathrm{i}y,x-\mathrm{i}y) \quad (6.15)$$

and

$$\frac{1}{2}\left(x^2+y^2-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)l_{m,n}(x+\mathrm{i}y,x-\mathrm{i}y)=(m+n+1)l_{m,n}(x+\mathrm{i}y,x-\mathrm{i}y). \quad (6.16)$$

This is the real representation of the eigenvalue equation for a two-dimensional harmonic oscillator. The Laguerre 2D-functions and the Hermite 2D-functions form only two different bases in the realization of the Heisenberg–Weyl algebra $w(2,\mathrm{R})$ by functions of two variables (z,z^*) or (x,y). They are related to each other as the basis of two opposite circular polarized modes to the basis of two perpendicular linearly polarized modes. This suggests that it should be possible to introduce a more general set of 2D-functions which makes the continuous transition from the Laguerre 2D-functions to the Hermite 2D-functions for a two-dimensional degenerate harmonic oscillator and which corresponds to two opposite elliptically polarized modes. We do not try to make this in the present paper.

7. Fock-state representation of the quasi-probabilities by Laguerre 2D-functions

We now consider applications of the Laguerre 2D-functions in quantum optics. By using this set of functions, the class of quasi-probabilities $F_{(0,0,r)}(\alpha,\alpha^*)$ with the ordering parameter r [29] can be represented in the following way

$$F_{(0,0,r)}(\alpha,\alpha^*)=\frac{2}{\sqrt{\pi}(1+r)}\exp\left(\frac{2r\,\alpha\alpha^*}{1-r^2}\right)\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle m|\varrho|n\rangle\left(\sqrt{\frac{1-r}{1+r}}\right)^{m+n}l_{n,m}\left(\frac{2\alpha}{\sqrt{1-r^2}},\frac{2\alpha^*}{\sqrt{1-r^2}}\right). \quad (7.1)$$

If we insert the explicit form of the Laguerre 2D-functions given in (3.2) and (3.3), we come to a known form of these quasi-probabilities and although we do not give the derivations here [3] (see also [1]). We mention here that for $r=0$, we get the Wigner quasi-probability $W(\alpha,\alpha^*)$, for $r=1$ the coherent-state (or Husimi–Kano) quasi-probability $Q(\alpha,\alpha^*)$ and for $r=-1$ the most singular Glauber–Sudarshan quasi-probability $P(\alpha,\alpha^*)$. For both cases $r=\pm1$ they have to be considered as limiting cases in (7.1), where the case $r=+1$ is unproblematic contrary to the case where $r=-1$.

The orthonormality (3.9) and completeness (3.10) of the Laguerre 2D-functions provide immediately the inversion of (7.1) that means the Fock-state matrix elements determined by the quasi-probabilities

$$\langle m|\varrho|n\rangle=\frac{2\sqrt{\pi}}{1-r}\left(\sqrt{\frac{1+r}{1-r}}\right)^{m+n}\int\frac{\mathrm{i}}{2}\,\mathrm{d}\alpha\wedge\mathrm{d}\alpha^*F_{(0,0,r)}(\alpha,\alpha^*)\times\exp\left(-\frac{2r\,\alpha\alpha^*}{1-r^2}\right)l_{m,n}\left(\frac{2\alpha}{\sqrt{1-r^2}},\frac{2\alpha^*}{\sqrt{1-r^2}}\right). \quad (7.2)$$

In the special case of the Wigner quasi-probability $r=0$, one obtains

$$W(\alpha,\alpha^*)=\frac{2}{\sqrt{\pi}}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle m|\varrho|n\rangle\,l_{n,m}(2\alpha,2\alpha^*) \quad (7.3)$$

with the inversion

$$\langle m|\varrho|n\rangle = 2\sqrt{\pi}\int \frac{\mathrm{i}}{2}\,\mathrm{d}\alpha\wedge\mathrm{d}\alpha^* W(\alpha,\alpha^*) l_{m,n}(2\alpha,2\alpha^*). \tag{7.4}$$

This shows that the Wigner quasi-probability $W(\alpha,\alpha^*)$ in Fock-state representation is almost directly determined by the set of Laguerre 2D-functions with the Fock-state matrix elements as coefficients.

The Glauber–Sudarshan quasi-probability $P(\alpha,\alpha^*)$ corresponding to $r=-1$ can be represented by the following limiting procedure by setting $r=-\sqrt{1-4\varepsilon}$, $\Rightarrow \varepsilon=(1-r^2)/4$ in (7.1)

$$P(\alpha,\alpha^*) = \lim_{\varepsilon\to 0}\left\{\frac{1}{\sqrt{\pi}\,\varepsilon}\exp\left(\alpha\alpha^* - \frac{\alpha\alpha^*}{2\varepsilon}\right)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\langle m|\varrho|n\rangle}{(\sqrt{\varepsilon})^{m+n}} l_{n,m}\left(\frac{\alpha}{\sqrt{\varepsilon}},\frac{\alpha^*}{\sqrt{\varepsilon}}\right)\right\} \tag{7.5}$$

that leads to a superposition of two-dimensional delta functions and their derivatives [19]. Peřina and Mišta [21, 22] (see also [23] and [24, 25]) introduced a 'regularized' representation of the Glauber–Sudarshan quasi-probability $P(\alpha,\alpha^*)$ by Laguerre polynomials. It is of a similar structure as (7.5) without a limiting procedure but therefore with changed matrix elements as coefficients. It can be represented in a favourable way by means of the Laguerre 2D-functions where their orthonormality properties can be used for its derivation. In this way, one finds the following modified representation of the Peřina–Mišta representation [1]

$$P(\alpha,\alpha^*) = \frac{1}{\sqrt{\pi}\,\varepsilon}\exp\left(\alpha\alpha^* - \frac{\alpha\alpha^*}{2\varepsilon}\right)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\varrho_{m,n}(\varepsilon)}{(\sqrt{\varepsilon})^{m+n}}\, l_{n,m}\left(\frac{\alpha}{\sqrt{\varepsilon}},\frac{\alpha^*}{\sqrt{\varepsilon}}\right) \tag{7.6}$$

with the following definition of the new matrix elements $\varrho_{m,n}(\varepsilon)$ together with their inversion (found by Peřina and coworkers [24, 25] and represented in the present form in [1])

$$\begin{aligned}\varrho_{m,n}(\varepsilon) &\equiv \sum_{k=0}^{\{m,n\}}\frac{\sqrt{m!n!}\,(-\varepsilon)^k}{k!\sqrt{(m-k)!(n-k)!}}\langle m-k|\varrho|n-k\rangle\\ \langle m|\varrho|n\rangle &= \sum_{l=0}^{\{m,n\}}\frac{\sqrt{m!n!}\,\varepsilon^l}{l!\sqrt{(m-l)!(n-l)!}}\varrho_{m-l,n-l}(\varepsilon).\end{aligned} \tag{7.7}$$

We have represented the quasi-probabilities up to now by complex coordinates (α,α^*). These complex coordinates (α,α^*) are connected with the canonical coordinates (q,p) in quantum optics by

$$\alpha = \frac{q+\mathrm{i}p}{\sqrt{2\hbar}} \equiv \frac{z}{\sqrt{2\hbar}} \qquad \alpha^* = \frac{q-\mathrm{i}p}{\sqrt{2\hbar}} \equiv \frac{z^*}{\sqrt{2\hbar}}. \tag{7.8}$$

This means that we identify the real coordinates (x,y) in the general treatment with the canonical coordinates (q,p) in quantum optics. By inserting the connection (6.4) or (6.5) between Laguerre 2D-functions and Hermite 2D-functions into (7.1), one obtains a new representation for the quasi-probabilities $F_{(0,0,r)}(q,p)$ in canonical coordinates (q,p). Taking into account the normalization of these quasi-probabilities with the integration measure $\mathrm{d}q\wedge\mathrm{d}p$ and the connection $(\mathrm{i}/2)\,\mathrm{d}\alpha\wedge\mathrm{d}\alpha^* = \mathrm{d}q\wedge\mathrm{d}p/(2\hbar)$ one obtains for the representation in canonical coordinates

$$\begin{aligned}F_{(0,0,r)}(q,p) &= \frac{1}{\hbar\pi(1+r)}\exp\left(-\frac{q^2+p^2}{\hbar(1+r)}\right)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle m|\varrho|n\rangle\left(\sqrt{\frac{1-r}{1+r}}\right)^{m+n}\frac{1}{2^{m+n}\sqrt{m!n!}}\\ &\quad\times\sum_{j=0}^{m+n}(\mathrm{i}2)^j P_j^{(n-j,m-j)}(0) H_{m+n-j}\left(\sqrt{\frac{2}{\hbar(1-r^2)}}q\right)H_j\left(\sqrt{\frac{2}{\hbar(1-r^2)}}p\right)\end{aligned} \tag{7.9}$$

and, in particular, for the Wigner quasi-probability $W(q,p)$ corresponding to $r=0$

$$W(q,p)=\frac{1}{\hbar\pi}\exp\left(-\frac{q^2+p^2}{\hbar}\right)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle m|\varrho|n\rangle\frac{1}{2^{m+n}\sqrt{m!n!}}$$
$$\times\sum_{j=0}^{m+n}(\mathrm{i}2)^j P_j^{(n-j,m-j)}(0)H_{m+n-j}\left(\sqrt{\frac{2}{\hbar}}q\right)H_j\left(\sqrt{\frac{2}{\hbar}}p\right). \quad (7.10)$$

The special case of this representation of the Wigner quasi-probability for a Fock state $\varrho=|n\rangle\langle n|$ which can be directly obtained from its representation by Laguerre polynomials by using the known relation (6.10) possesses in representation by the Hermite polynomials the form

$$\varrho=|n\rangle\langle n|\Leftrightarrow W(q,p)=\frac{1}{\hbar\pi}\exp\left(-\frac{q^2+p^2}{\hbar}\right)\frac{1}{2^{2n}n!}$$
$$\times\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}H_{2n-2k}\left(\sqrt{\frac{2}{\hbar}}q\right)H_{2k}\left(\sqrt{\frac{2}{\hbar}}p\right) \quad (7.11)$$

which is derived in [26] (see also [17], equation (3.51)). The inversion formulae (7.3) and (7.4) can be represented in an analogous way by Hermite polynomials instead of Laguerre 2D-functions.

The above considerations show that the Laguerre 2D-functions are very appropriate for the representation of the quasi-probabilities in the Fock-state basis and for their inversion.

8. Radon transform of the Wigner quasi-probability

The Radon transform of the Wigner quasi-probability or of a more smoothed quasi-probability is the main object of quantum tomography for the reconstruction of states by means of the Wigner quasi-probability or of the density operator from measured quadrature distributions [27, 28] (see also [10]). The Radon transform $\breve{W}(u,v;c)$ of the Wigner quasi-probability $W(q,p)$ in canonical coordinates is defined by [6]

$$\breve{W}(u,v;c)\equiv\int \mathrm{d}q\wedge\mathrm{d}p\,\delta(c-uq-vp)W(q,p) \quad (8.1)$$

and it is connected with the rotated quadrature components considered in section 2 in the following way

$$\breve{W}(\cos\varphi,\sin\varphi;q)=\langle q;\varphi|\varrho|q;\varphi\rangle. \quad (8.2)$$

The Fock-state representation of the Wigner quasi-probability (7.3) written by means of the complex variables (z,z^*) according to (7.8) and normalized with the measure $(\mathrm{i}/2)\,\mathrm{d}z\wedge\mathrm{d}z^*$ instead of $(\mathrm{i}/2)\,\mathrm{d}\alpha\wedge\mathrm{d}\alpha^*$ is now

$$\frac{1}{2\hbar}W\left(\frac{z}{\sqrt{2\hbar}},\frac{z^*}{\sqrt{2\hbar}}\right)=\frac{1}{\hbar\sqrt{\pi}}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle m|\varrho|n\rangle l_{n,m}\left(\sqrt{\frac{2}{\hbar}}z,\sqrt{\frac{2}{\hbar}}z^*\right). \quad (8.3)$$

We have already calculated the Radon transform of the Laguerre 2D-functions in equations (4.20) or (4.21). Taking into account the stretch factor according to (4.11), the Radon transform of this function is

$$\breve{W}(w,w^*;c)\equiv\frac{1}{2\hbar}\int\frac{\mathrm{i}}{2}\,\mathrm{d}z\wedge\mathrm{d}z^*\,\delta\left(c-\frac{1}{2}\left(w^*z+wz^*\right)\right)W\left(\frac{z}{\sqrt{2\hbar}},\frac{z^*}{\sqrt{2\hbar}}\right)$$
$$=\frac{1}{\sqrt{\hbar ww^*}}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle m|\varrho|n\rangle\left(\sqrt{\frac{w}{w^*}}\right)^{n-m}h_m\left(\frac{c}{\sqrt{\hbar ww^*}}\right)h_n\left(\frac{c}{\sqrt{\hbar ww^*}}\right). \quad (8.4)$$

Expressed by means of the Hermite polynomials, this takes on the form

$$\breve{W}(w, w^*; c) = \frac{1}{\sqrt{\pi\hbar w w^*}} \exp\left(-\frac{c^2}{\hbar w w^*}\right) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \langle m|\varrho|n\rangle \times \left(\sqrt{\frac{w}{w^*}}\right)^{n-m} \frac{1}{\sqrt{2^{m+n} m! n!}} H_m\left(\frac{c}{\sqrt{\hbar w w^*}}\right) H_n\left(\frac{c}{\sqrt{\hbar w w^*}}\right). \quad (8.5)$$

In this form, we can set $w = u + \mathrm{i}v$ and $w^* = u - \mathrm{i}v$ and we obtain immediately by this substitution the real representation $\breve{W}(u, v; c)$ of the Radon transform of the Wigner quasi-probability $W(q, p)$ in canonical coordinates (q, p). From this form, one finds $\breve{W}(\cos\varphi, \sin\varphi; q)$ by specialization (cf equation (2.10)), however, it is more effective to possess the form $\breve{W}(u, v; c)$ or $\breve{W}(w, w^*; c)$ of the Radon transform of the Wigner quasi-probability because displacement and squeezing transformations of the states lead to simple transformations of the arguments $(u, v; c)$ or $(w, w^*; c)$ of the Radon transform and it is not necessary to make the more difficult calculations for these states anew. In particular, squeezing of the states leads to symplectic transformations of the variables (u, v) or (w, w^*) which do not preserve the angles. These transformations are explicitly given in [6] together with an example. Therefore, it is irrational to work from the beginning with the specialization $\breve{W}(\cos\varphi, \sin\varphi; q)$ in which the angles explicitly appear and which is difficult to transform. On the other side, it is easy to obtain $\breve{W}(\cos\varphi, \sin\varphi; q)$ from the general $\breve{W}(u, v; c)$ by using the homogeneity condition (4.4) after division of the arguments by $u^2 + v^2 = ww^*$, whereas the inverse transition is not as simple to carry out.

9. Fock-state representation of the displacement operator and scalar product of displaced Fock states

The unitary displacement operator $D(\alpha, \alpha^*)$ as the essential part of the general element of the Heisenberg–Weyl group $W(1, \mathrm{R})$ is defined by [2]

$$D(\alpha, \alpha^*) \equiv \exp(\alpha a^\dagger - \alpha^* a). \quad (9.1)$$

The matrix elements of this displacement operator are almost directly given by the Laguerre 2D-functions [8]. We do not calculate them here (e.g., [3, 8]) but mention the following. The action of the displacement operator $D(\alpha, \alpha^*)$ onto Fock states $|n\rangle$ provides by definition the displaced Fock states $|\alpha, n\rangle$ (e.g, [11])

$$|\alpha, n\rangle \equiv D(\alpha, \alpha^*)|n\rangle \qquad \langle\alpha, n| \equiv \langle n|(D(\alpha, \alpha^*))^\dagger. \quad (9.2)$$

The scalar product of displaced Fock states is calculated in [11] (equation (3.11) there) and takes on the following form in representation by the Laguerre 2D-functions

$$\langle\beta, m|\alpha, n\rangle = (-1)^n \sqrt{\pi} \exp\{\tfrac{1}{2}(\alpha\beta^* - \alpha^*\beta)\} l_{m,n}(\alpha - \beta, \alpha^* - \beta^*). \quad (9.3)$$

The matrix element $\langle m|D(\alpha, \alpha^*)|n\rangle$ of the displacement operator is the special case $\beta = 0$ of the general scalar product of displaced Fock states $\langle\beta, m|\alpha, n\rangle$. Therefore, one obtains for these matrix elements

$$\langle m|D(\alpha, \alpha^*)|n\rangle = \langle m|\alpha, n\rangle = (-1)^n \sqrt{\pi} l_{m,n}(\alpha, \alpha^*). \quad (9.4)$$

The displacement operator $D(\alpha, \alpha^*)$ is connected with the Heisenberg–Weyl group $W(1, \mathrm{R})$, whereas the Laguerre 2D-functions are simply connected with the Heisenberg–Weyl group $W(2, \mathrm{R})$ as was shown in sections 5 and 6. Nevertheless, the Laguerre 2D-functions can often be applied to problems which are not directly connected with the last mentioned group.

10. Conclusion

We have derived basic relations for a two-dimensional orthonormalized set of functions $l_{m,n}(z, z^*)$ which is called the set of Laguerre 2D-functions and have applied this to the quasi-probabilities in Fock-state representation. The derived relations include the Fourier and Radon transform of the Laguerre 2D-functions. It was shown that the Laguerre 2D-functions are eigensolutions of the differential equation for a two-dimensional degenerate harmonic oscillator (equal frequencies) and that they form a basis for a realization of the Heisenberg–Weyl algebra $w(2, \mathrm{R})$. Further relations are established between Laguerre 2D-functions and superpositions of products of two Hermite functions and their inversion was found. These are relations between the different bases of circular and linear polarization of a two-mode system. Due to the orthonormalization and completeness of the Laguerre 2D-functions they find interesting applications in problems with two or more variables. We considered here applications in quantum optics to the two-dimensional phase space of one mode.

Appendix A. Radon transform of Laguerre 2D-functions

We present in this appendix the calculation of the Radon transform of the Laguerre 2D-functions.

First, we evaluate auxiliary integrals which lead to Hermite polynomials as follows

$$\begin{aligned}
&\frac{1}{\sqrt{\pi}a}\int_{-\infty}^{+\infty} \mathrm{d}x\ \exp\left(-\frac{(x-x_0)^2}{a^2}\right)x^n \\
&\quad = \frac{1}{\sqrt{\pi}a}\int_{-\infty}^{+\infty} \mathrm{d}y\ \exp\left(-\frac{y^2}{a^2}\right)(x_0+y)^n \\
&\quad = \sum_{l=0}^{[n/2]} \frac{n!}{(2l)!(n-2l)!} x_0^{n-2l} \frac{1}{\sqrt{\pi}a}\int_{-\infty}^{+\infty} \mathrm{d}y\ \exp\left(-\frac{y^2}{a^2}\right) y^{2l} \\
&\quad = \sum_{l=0}^{[n/2]} \frac{n!}{l!(n-2l)!} x_0^{n-2l}\left(\frac{a}{2}\right)^{2l} \\
&\quad = \left(\mp\mathrm{i}\frac{a}{2}\right)^n H_n\left(\pm\mathrm{i}\frac{x_0}{a}\right).
\end{aligned} \tag{A.1}$$

Now, by applying these integrals, one obtains the Radon transform of the Laguerre 2D-functions by the following chain of identities

$$\begin{aligned}
\check{l}_{m,n}(w, w^*; c) &= 2\pi(-\mathrm{i})^{m+n}\frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}b\, l_{m,n}(bw, bw^*)\exp(\mathrm{i}bc) \\
&= \frac{1}{\sqrt{\pi}}\exp\left(-\frac{c^2}{2ww^*}\right)\frac{(-\mathrm{i})^{m+n}}{\sqrt{m!n!}}\sum_{j=0}^{\{m,n\}}\frac{m!n!(-1)^j}{j!(m-j)!(n-j)!}w^{m-j}w^{*\,n-j} \\
&\quad\times\int_{-\infty}^{+\infty} \mathrm{d}b\ \exp\left\{-\frac{ww^*}{2}\left(b-\mathrm{i}\frac{c}{ww^*}\right)^2\right\}b^{m+n-2j} \\
&= \sqrt{\frac{2}{ww^*}}\exp\left(-\frac{c^2}{2ww^*}\right)\left(\sqrt{\frac{w}{w^*}}\right)^{m-n} \\
&\quad\times\frac{1}{\sqrt{2^{m+n}m!n!}}\sum_{j=0}^{\{m,n\}}\frac{m!n!2^j}{j!(m-j)!(n-j)!}H_{m+n-2j}\left(\frac{c}{\sqrt{2ww^*}}\right)
\end{aligned}$$

$$= \sqrt{\frac{2}{ww^*}} \exp\left(-\frac{c^2}{2ww^*}\right) \left(\sqrt{\frac{w}{w^*}}\right)^{m-n} \frac{1}{\sqrt{2^{m+n} m!n!}}$$
$$\times H_m\left(\frac{c}{\sqrt{2ww^*}}\right) H_n\left(\frac{c}{\sqrt{2ww^*}}\right). \tag{A.2}$$

This can be written by means of Hermite functions as follows

$$\breve{l}_{m,n}(w, w^*; c) = \sqrt{\frac{2\pi}{ww^*}} \left(\sqrt{\frac{w}{w^*}}\right)^{m-n} h_m\left(\frac{c}{\sqrt{2ww^*}}\right) h_n\left(\frac{c}{\sqrt{2ww^*}}\right). \tag{A.3}$$

This Radon transform possesses a simple product structure with factors depending on m and n but without coupling terms between them.

Appendix B. Relations between products of powers of the real and complex variables

The relations between the products of power functions of the real variables (x, y) and of the complex variables (z, z^*) are easy to obtain and to represent in a compact manner if one only looks to the explicit form of Jacobi polynomials for a possible representation of the arising coefficients. In preparation of this, we write down the following explicit representation of Jacobi polynomials $P_j^{(\alpha,\beta)}(u)$ [9]

$$P_j^{(\alpha,\beta)}(u) = \left(\frac{u-1}{2}\right)^j \sum_{k=0}^{j} \frac{(j+\alpha)!(j+\beta)!}{k!(j-k)!(j+\alpha-k)!(\beta+k)!} \left(\frac{u+1}{u-1}\right)^k. \tag{B.1}$$

Now, by applying the binomial formula to powers of $x \pm \mathrm{i}y$, one quickly proceeds to the following identity

$$z^m z^{*n} = (x+\mathrm{i}y)^m (x-\mathrm{i}y)^n$$
$$= \sum_{j=0}^{m+n} x^{m+n-j} y^j (-\mathrm{i})^j \sum_{k=0}^{j} \frac{m!n!}{k!(j-k)!(m-k)!(n-j+k)!} (-1)^k. \tag{B.2}$$

By comparison of the inner sum with the representation (B.1) of Jacobi polynomials, one finds that it can be represented by them by choosing $u = 0$ for the argument and by choosing $\alpha = m - j$ and $\beta = n - j$. In this way, one obtains

$$z^m z^{*n} = \sum_{j=0}^{m+n} (\mathrm{i}2)^j P_j^{(m-j,n-j)}(0) x^{m+n-j} y^j. \tag{B.3}$$

In a fully analogous way, one finds its inversion

$$x^m y^n = \frac{(-\mathrm{i})^n}{2^{m+n}} \sum_{j=0}^{m+n} 2^j P_j^{(m-j,n-j)}(0) z^{m+n-j} z^{*j}. \tag{B.4}$$

The Jacobi polynomials, in particular, for zero argument possess many transformation relations and one can give the relations (B.3) and (B.4) many slightly different forms but the chosen form is one of the most symmetrical ones. The derivations of (B.3) and (B.4) are simple if one only examines the Jacobi polynomials for a possible representation. It appears that the resulting formulae do not exist in printed form since we cannot give any reference for them, therefore, we consider here the derivation.

It is not possible to derive simple closed formulae for the result of the sums in the Jacobi polynomials for zero argument and without specialization of the indices (j, α, β) in $P_j^{(\alpha,\beta)}(0)$. From their meaning in (B.3), it is clear that $2^j P_j^{(m-j,n-j)}(0)$ with $j =$

$0, 1, \ldots, (m+n)$ for given (m, n) are integers which vanish for $j < 0$ and $j > m+n$. We decomposed $2^j P_j^{(m-j,n-j)}(0)$ for many fixed pairs (m, n) into prime numbers where for low values of (m, n) and $0 \leqslant j \leqslant m+n$ there irregularly appear high prime factors that makes a simple formula for this sum impossible. The only values of the arguments where the sum in the Jacobi polynomials can be generally evaluated in a closed way are $u = \pm 1$. In all other cases one gets simple formulae for this sum if additional relations between the indices in the Jacobi polynomials are considered and, seemingly, many interesting identities are unknown up to now. Such a case is equal upper indices for zero argument and the corresponding simplifications are given in (6.6). For equal upper indices, the Jacobi polynomials are related to certain Gegenbauer polynomials and to associated Legendre polynomials [14]. Knowing the relations (B.3) and (B.4), it becomes immediately clear that the Jacobi polynomials for zero argument $P_j^{(m-j,n-j)}(0)$ should give simple closed expressions for $m = n$ because the relations $(zz^*)^n = (x^2 + y^2)^n$ can be expanded by using binomial formula.

We mention here that similar relations of the form (B.3) and (B.4) can be derived for the relations between symmetrically ordered powers of the boson operators $(a, a^\dagger)$ and of the canonical operators (Q, P) with the basic relations between these pairs of operators given in (2.6) [29]. They are ($\mathcal{S}\{\ldots\}$ means symmetrical ordering of content in braces)

$$\mathcal{S}\{a^m a^{\dagger n}\} = \frac{1}{(\sqrt{2\hbar})^{m+n}} \sum_{j=0}^{m+n} (\mathrm{i}2)^j P_j^{(m-j,n-j)}(0) \mathcal{S}\{Q^{m+n-j} P^j\}. \tag{B.5}$$

and

$$\mathcal{S}\{Q^m P^n\} = (-\mathrm{i})^n \left(\sqrt{\frac{\hbar}{2}}\right)^{m+n} \sum_{j=0}^{m+n} 2^j P_j^{(m-j,n-j)}(0) \mathcal{S}\{a^{m+n-j} a^{\dagger j}\}. \tag{B.6}$$

The composition of the two identities (B.3) and (B.4) or (B.5) and (B.6) leads to the following identity for the Jacobi polynomials at zero argument checked by computer

$$\frac{1}{2^{m+n}} \sum_{j=0}^{m+n} 2^{j+k} P_j^{(m-j,n-j)}(0) P_k^{(m+n-j-k,j-k)}(0) = \delta_{k,n}. \tag{B.7}$$

This identity can be used for the inversion of formulae of type (B.3) or (B.4).

References

[1] Wünsche A 1998 *Acta Phys. Slovaca* **48**

[2] Glauber R 1963 *Phys. Rev.* **131** 2766

[3] Cahill K E and Glauber R J 1969 *Phys. Rev.* **177** 1857
Cahill K E and Glauber R J 1969 *Phys. Rev.* **177** 1882

[4] Gel'fand I M, Grayev M I and Vilenkin N Y 1966 *Generalized Functions, Vol. 5 Integral Geometry and Representation Theory* (New York: Academic)

[5] Barrett H H 1998 *Progr. Opt.* **21** 217

[6] Wünsche A 1997 *J. Mod. Opt.* **44** 2293

[7] Weyl H 1931 *Gruppentheorie und Quantenmechanik* (Engl. Transl. 1931 *The Theory of Groups and Quantum Mechanics* (New York: Dover))

[8] Perelomov A 1986 *Generalized Coherent States and Their Application* (Berlin: Springer)

[9] Erdélyi A 1953 *Higher Transcendental Functions* vol II (New York: McGraw-Hill)

[10] Leonhardt U 1997 *Measuring the Quantum State of Light* (Cambridge: Cambridge University Press)

[11] Wünsche A 1991 *Quantum Opt.* **3** 359

[12] Dodonov V V and Man'ko V I 1994 *J. Math. Phys.* **35** 4277

[13] Dodonov V V 1994 *J. Phys. A: Math. Gen.* **27** 6191

[14] Man'ko V I and Wünsche A 1997 *Quantum Semiclass. Opt.* **9** 381

[15] Lebedev N N 1965 *Special Functions and Their Applications* (Englewood Cliffs, NY: Prentice Hall)

[16] Gel'fand I M and Shilov G E 1964 *Generalized Functions Vol. 1 Properties and Operations* (Boston, MA: Academic)

[17] Kim Y S and Noz M E 1991 *Phase Space Picture of Quantum Mechanics* (Singapore: World Scientific)

[18] Campos R A, Saleh B E A and Teich M C 1989 *Phys. Rev.* A **40** 1371

[19] Wünsche A 1996 *Quantum Semicl. Opt.* **8** 343

[20] Hong-yi Fan, Zu-wei Liu and Tu-nan Ruan 1985 *Commun. Theor. Phys. (Beijing)* **3** 175 (I acquired knowledge of this paper from Professor A Vourdas from Liverpool some years ago in connection with the solution of the eigenvalue problem of the boson creation operator which I published in 1992 independently of this paper in [30])

[21] Peřina J and Mišta L 1968 *Phys. Lett.* **27A** 217

[22] Peřina J and Mišta L 1969 *Ann. Phys.* **22** 372

[23] Lukš A 1976 *Czech. J. Phys.* B **26** 1095

[24] Peřina J 1971 *Coherence of Light* (London: Van Nostrand)

[25] Peřina J 1991 *Quantum Statistics of Linear and Nonlinear Optical Phenomena* 2nd edn (Dordrecht: Kluwer)

[26] Schleich W, Walls D C and Wheeler J A 1988 *Phys. Rev.* A **38** 1177

[27] Vogel K and Risken H 1989 *Phys. Rev.* A **40** 2847

[28] Raymer M G, Smithey D T, Beck M and Cooper J 1994 *Acta Phys. Pol.* A **86** 71

[29] Wünsche A 1998 *Quantum Semicl. Opt.* submitted

[30] Wünsche A 1992 *Ann. Phys. Lpz.* **1** 181